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VIBRATION OF A CANTILEVER BEAM THAT SLIDES AXIALLY IN A RIGID F--ETC(U)

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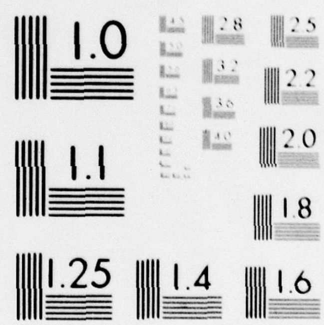
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VIBRATION OF A CANTILEVER BEAM THAT

SLIDES AXIALLY IN A RIGID FRICTIONLESS HOLE

by

Arthur P. Boresi and
David Salinas

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This paper considers a cantilever beam which can move in and out of a rigid frictionless hole. Hamilton's principle is used for the development of two governing equations, and boundary conditions defined over a moving boundary. A transformation of coordinates is used to transform the problem into a system of two nonlinear p.d.e. defined over a fixed region. A discussion of some mathematical and physical considerations is presented. Finally, a finite element formulation of the problem is presented.

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VIBRATION OF A CANTILEVER BEAM THAT

SLIDES AXIALLY IN A RIGID FRICTIONLESS HOLE

INTRODUCTION

Consider a cantilever beam that fits snugly into a frictionless hole, Fig. 1. Axes x,y are fixed. A force F(t) moves the beam axially. Hence, the length L of the beam outside of the hole is a function of time t. The lateral deflection of the beam is y(x,t), where for x < 0, y = 0. The axial velocity of the beam is $\dot{L} = dL/dt$.

By Hamilton's principle,

$$\int_0^{t_1} (\delta T + \delta W)\, dt = 0 \tag{1}$$

where T denotes the kinetic energy of the system and W denotes the work of external and internal forces. The kinetic energy is

$$T = \frac{1}{2}\rho L_o \dot{L}^2 + \frac{1}{2}\rho \int_0^L y_t^{\,2}\, dx \tag{2}$$

where the subscript t denotes partial differentiation, ρ = the mass of the beam per unit length = constant, and L_o = the length of the beam when t = 0. The strain energy of the beam (the negative of the work of internal forces for a conservative system) is

$$U = \frac{1}{2} EI \int_0^L y_{xx}^2 \, dx \tag{3}$$

where E = the modulus of elasticity of the beam and I = the second moment of the beam cross-sectional area. Hence, the virtual work δW is

$$\delta W = F\delta L - \delta U \tag{4}$$

where δU = the first variation of the strain energy (Eq. 3). Because of the inertia, the axial movement of the beam tends to bend the beam by beam-column action. This effect is neglected here, but it could be included.

Equations (1), (2), (3), and (4) yield

$$\int_0^{t_1} [\rho L_0 \dot{L} \delta \dot{L} + \frac{1}{2} \rho y_t^{\,2}(L,t) \, \delta L + F\delta L - \frac{1}{2} EI y_{xx}^2(L,t) \, \delta L$$

$$+ \rho \int_0^L y_t \delta y_t dx - EI \int_0^L y_{xx} \delta y_{xx} dx] \, dt = 0 \tag{5}$$

If L(t) is considered as given, the terms with the factors δL and $\delta \dot{L}$ merely determine F(t) to give the prescribed motion L(t). The Euler equation for the lateral motion of the beam is

$$EI y_{xxxx} + \rho y_{tt} = 0 \tag{6}$$

The natural boundary conditions are

$$y_{xx}(L,t) = y_{xxx}(L,t) = 0 \tag{7}$$

The forced boundary conditions are

$$y(o,t) = y_x(o,t) = 0 \tag{8}$$

Adopting Hamilton's viewpoint, we may suppose that $y(x,o)$ and $y(x,t_1)$ are prescribed (Fig. 2). The problem is to determine $y(x,t)$ in region R (Fig. 2) subject to the differential equation (Eq. 6) and the indicated boundary conditions (Fig. 2).

The curved boundary at the right side of region R is troublesome. It is possible to transform coordinates so that this boundary becomes straight.

Let

$$x = x(\xi,\eta), \; t = t(\xi,\eta) \tag{9}$$

Regard y as a function of (ξ,η). Then,

$$y = y(\xi,\eta), \; \xi = \xi(x,t), \; \eta = \eta(x,t) \tag{10}$$

and

$$\begin{aligned}
\left(\frac{\partial y}{\partial x}\right)_t &= \left(\frac{\partial y}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial x}\right)_t + \left(\frac{\partial y}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial x}\right)_t \\
\left(\frac{\partial y}{\partial t}\right)_x &= \left(\frac{\partial y}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial t}\right)_x + \left(\frac{\partial y}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial t}\right)_x = \dot{y} \\
\left(\frac{\partial^2 y}{\partial t^2}\right)_x &= \left(\frac{\partial \dot{y}}{\partial \xi}\right)_\eta \left(\frac{\partial \xi}{\partial t}\right)_x + \left(\frac{\partial \dot{y}}{\partial \eta}\right)_\xi \left(\frac{\partial \eta}{\partial t}\right)_x
\end{aligned} \tag{11}$$

Here, we regard $\dot{y}$ as being a function of (ξ,η).

Now let

$$\xi = \frac{x}{L}, \; t = \eta \tag{12}$$

When we write $\frac{\partial y}{\partial \xi}$, $\frac{\partial y}{\partial \eta}$, etc., we imply $y = y(\xi,\eta)$. Hence, $\frac{\partial y}{\partial x} = (\frac{\partial y}{\partial \xi})(\frac{1}{L})$, and similarly,

$$\frac{\partial^4 y}{\partial x^4} = \frac{1}{L^4} \cdot \frac{\partial^4 y}{\partial \xi^4} \tag{13}$$

Now,

$$(\frac{\partial y}{\partial t})_x = \frac{\partial y}{\partial \xi} (\frac{\partial \xi}{\partial t})_x + \frac{\partial y}{\partial \eta} = \dot{y}$$

and since

$$(\frac{\partial \xi}{\partial t})_x = - \frac{x\dot{L}}{L^2} = - \frac{\xi \dot{L}}{L} , \ (\frac{\partial \eta}{\partial t})_x = 1, \tag{14}$$

we find

$$\dot{y} = -(\frac{\partial y}{\partial \xi})_\eta \frac{\xi \dot{L}}{L} + (\frac{\partial y}{\partial \eta})_\xi$$

and hence,

$$\begin{aligned}(\frac{\partial \dot{y}}{\partial \xi})_\eta &= -(\frac{\partial^2 y}{\partial \xi^2}) \frac{\xi \dot{L}}{L} - (\frac{\partial y}{\partial \xi})_\eta \frac{\dot{L}}{L} + \frac{\partial^2 y}{\partial \xi \partial \eta} \\ (\frac{\partial \dot{y}}{\partial \eta})_\xi &= - \frac{\partial^2 y}{\partial \xi \partial \eta} \frac{\xi \dot{L}}{L} - (\frac{\partial y}{\partial \xi})_\eta \frac{\xi \ddot{L}}{L} + (\frac{\partial y}{\partial \xi})_\eta \frac{\xi \dot{L}^2}{L} \\ &\quad + \frac{\partial^2 y}{\partial \eta^2}\end{aligned} \tag{15}$$

Thus, by the third of Eqs. (11), Eqs. (12), Eqs. (14), and Eqs. (15), we find

$$\begin{aligned}(\frac{\partial^2 y}{\partial t^2})_x &= [-(\frac{\partial^2 y}{\partial \xi^2}) \frac{\xi \dot{L}}{L} - (\frac{\partial y}{\partial \xi})\frac{\dot{L}}{L} + \frac{\partial^2 y}{\partial \xi \partial \eta}] \ (- \frac{\xi \dot{L}}{L}) \\ &\quad + [-(\frac{\partial^2 y}{\partial \xi \partial \eta})\frac{\xi \dot{L}}{L} - (\frac{\partial y}{\partial \xi})\frac{\xi \ddot{L}}{L} + (\frac{\partial y}{\partial \xi})\frac{\xi \dot{L}^2}{L^2} + \frac{\partial^2 y}{\partial \eta^2}]\end{aligned}$$

and since $\eta = t$, we obtain

$$\left(\frac{\partial^2 y}{\partial t^2}\right)_x = \frac{\partial^2 y}{\partial \xi^2}\frac{\xi^2 \dot{L}^2}{L^2} + 2\frac{\partial y}{\partial \xi}\frac{\xi \dot{L}^2}{L^2} - 2\frac{\partial^2 y}{\partial \xi \partial t}\frac{\xi \dot{L}}{L} - \frac{\partial y}{\partial \xi}\frac{\xi \ddot{L}}{L} + \frac{\partial^2 y}{\partial t^2} \tag{16}$$

in which $y = y(\xi, t)$ on the right hand side of Eq. (16).

Consequently, Eqs. (6), (13) and (16) yield

$$\frac{EI}{L^4}\frac{\partial^4 y}{\partial \xi^4} + \rho\left[\frac{\partial^2 y}{\partial \xi^2}\frac{\xi^2 \dot{L}^2}{L^2} + 2\frac{\partial y}{\partial \xi}\frac{\xi \dot{L}^2}{L^2} - 2\frac{\partial^2 y}{\partial \xi \partial t}\frac{\xi \dot{L}}{L} - \frac{\partial y}{\partial \xi}\frac{\xi \ddot{L}}{L} + \frac{\partial^2 y}{\partial t^2}\right] = 0 \tag{17}$$

where y is now regarded as a function of ξ and t. Equations (7) and (8) become

$$y_{\xi\xi}(1,t) = y_{\xi\xi\xi}(1,t) = 0$$
$$y\ (0,t) = y_{\xi}\ (0,t) = 0 \tag{18}$$

Equation (17) is much more complicated than Eq. (6). However, the region R is simpler (Fig. 3). The problem remains a linear boundary-value problem if in Eq. (17) we regard $L(t)$ as a given function of time t. If we consider $L(t)$ as unknown, we must obtain another equation which defines $L(t)$. This equation is obtained from Eq. (5) from the terms with the factors δL and $\delta \dot{L}$. Hence, we find

$$\ddot{L} + \frac{1}{2\rho L_o}[EIy_{xx}^2(L,t) - \rho y_t^{\,2}(L,t)] = \frac{1}{\rho L_o} F(t) \tag{19}$$

In terms of (ξ,t), Eq. (19) becomes

$$\begin{aligned} \ddot{L} &+ \frac{1}{2\rho L_o}\{\frac{EI}{L^4} y_{\xi\xi}^2(1,t) - \rho[-\frac{\dot{L}}{L} y_\xi(1,t) + y_t(1,t)]^2\} \\ &= \frac{1}{\rho L_o} F(t) \end{aligned} \tag{20}$$

Equation (20) is subject to the initial conditions

$$\begin{aligned} L(0) &= L_o \\ \dot{L}(0) &= 0 \end{aligned} \tag{21}$$

where L_o is the initial length of the beam at time $t = 0$, (see Fig. 1).

Thus, if $y(x,t)$ and $L(t)$ are considered as the unknowns, Eqs. (17) and (20) subject to the boundary conditions of Eqs. (18) and (21), respectively, represent a complicated nonlinear boundary value problem.

ANALYSIS OF EQUATIONS

(a) $L(t)$ Prescribed. If $L(t)$ is given, then the problem reduces to seeking a solution of Eq. (17) subject to the boundary conditions, Eqs. (18). The problem is still of considerable difficulty. However, the problem is a linear boundary-value problem. The nature of the solutions of Eq. (17) is unknown. Indeed, one must consider whether or not a unique solution of the boundary-value problem exists. Also, is there a unique solution of the corresponding initial value problem in which the condition $y = F_1(\xi)$ on $t = t_1$ is discarded and the functions $y(\xi,0)$ and $y_t(\xi,0)$ are prescribed? One way to approach such questions is through the calculus of finite

differences, in which the differential equation is replaced by a linear partial difference equation. The existence of a solution of the difference equation is a question of linear algebra. The most practical way of developing numerical solutions of Eqs. (17) and (18) is an open question. One possible approach is through the calculus of finite differences, or a combination of the calculus of finite differences with the Runge-Kutta method. Another approach is through finite-element methods.

The problem is simplified greatly if the axial motion is so slow that time derivatives of L(t) are negligible. For example, if the beam is vibrating in a natural mode and it is then drawn slowly into the hole, does it continue to vibrate in a natural mode? How do the frequency and amplitude vary with L under these conditions? For a rigid, frictionless hole, the total mechanical energy is constant, since no energy is imparted at the root. This criterion serves to determine how the amplitude varies. However, we should not have to apply this criterion, since constancy of mechanical energy should follow automatically from the solution of Eq. (17).

If $\dot{L}$ and $\ddot{L}$ are negligible, Eq. (17) reduces to

$$\frac{\partial^4 y}{\partial \xi^4} + \frac{\rho L^4}{EI} \frac{\partial^2 y}{\partial t^2} = 0 \tag{22}$$

Equation (22) is much simpler than Eq. (17). However, the fact that L is a prescribed function of t still complicates it. There is a question as to whether the character of the solution is altered by the reduction of Eq. (17) to Eq. (22). Again, does the initial value problem posed by Eqs. (18) and (22) have a unique solution? If so, the constancy of mechanical energy, as a separate principle, is not needed to determine the dependency of amplitude upon L. In fact, constancy of mechanical energy should be a deducible consequence of Eqs. (18) and (22).

Unlike Eq. (17), Eq. (22) allows a theory of natural modes. For example, consider solutions of Eq. (22) of the form $y = f(\xi)g(t)$. Then, Eq. (22) yields

$$\frac{f''''}{f} + \frac{\rho L^4}{EI}\frac{g''}{g} = 0$$

Hence,

$$\frac{f''''}{f} = \beta^4, \quad \frac{\rho L^4}{EI}\frac{g''}{g} = -\beta^4 \tag{23}$$

in which β is a constant. Then,

$$g'' + \frac{\beta^4 EI}{\rho L^4} g = 0 \tag{24}$$

Unfortunately, Eq. (24) does not yield simple harmonic motion, because L is a function of t. However, the equation

$$f'''' - \beta^4 f = 0 \tag{25}$$

can be integrated. With the root conditions (Eqs. 18) $f(o) = f'(o) = 0$, Eq. (25) yields

$$f(\xi) = A(\sinh\beta\xi - \sin\beta\xi) + B(\cosh\beta\xi - \cos\beta\xi) \tag{26}$$

Now, the free-end conditions (Eqs. 18) $f''(1) = f'''(1) = 0$ yield

$$\begin{aligned} A(\sinh\beta + \sin\beta) + B(\cosh\beta + \cos\beta) &= 0 \\ A(\cosh\beta + \cos\beta) + B(\sinh\beta - \sin\beta) &= 0 \end{aligned} \tag{27}$$

and the vanishing of the determinant of Eqs. (27) yields

$$\cosh\beta \cos\beta = -1 \tag{28}$$

These results are exactly like that for natural modes of a cantilever with fixed length. The roots of Eq. (28) are constants β_1, β_2, β_3, $\cdots$. They have been tabulated by Young and Felgar (1).

As is customary, let $B = 1$ and $A = -\alpha_n$. Then, by Eq. (27),

$$\alpha_n = \frac{\cosh\beta_n + \cos\beta_n}{\sinh\beta_n + \sin\beta_n} \tag{29}$$

Consequently, α_1, α_2, $\cdots$ are constants, independent of L. The values of α_n have been tabulated by Young and Felgar. Equation (26) gives the natural modes,

$$f_n(\xi) = \cosh\beta_n\xi - \cos\beta_n\xi - \alpha_n(\sinh\beta_n\xi - \sin\beta_n\xi) \tag{30}$$

The functions $f_n(\xi)$ are *orthogonal* in the interval (0,1); in fact, it is *easily shown that*

$$\begin{aligned}
&\int_0^1 f_n f_m d\xi = 0, \int_0^1 f_n'' f_m'' d\xi = 0, \ m \neq n \\
&\int_0^1 f_n^2 \, d\xi = 1, \int_0^1 (f_n'')^2 d\xi = \beta_n^4
\end{aligned} \tag{31}$$

The fact that the functions $f_n(\xi)$ satisfy the end conditions (Eq. 18), makes then ideal approximating functions for use in numerical attacks on Eqs. (17) and (22). Presumably, a solution of Eqs. (17) or (22) can be approximated by

$$y(\xi,t) = \sum_{n=1}^{N} q_n(t) f_n(\xi) \tag{32}$$

This approach reduces the problem to one of a finite number of degrees of freedom. The generalized coordinates are $q_n(t)$. The Lagrange theory of linear vibrations is then applicable. Accordingly, let

$$\omega_n = \frac{\beta_n^2}{L^2} \sqrt{\frac{EI}{\rho}} \tag{33}$$

Then, Eqs. (24) becomes

$$g_n'' + \omega_n^2 g_n = 0 \tag{34}$$

Since L depends on t, note that ω_n depends on t. Consequently, ω_n is not exactly the frequency of a harmonic motion. The general solution of Eq. (34) is

$$g_n = g_n(t, A_n, B_n) \tag{35}$$

in which A_n, B_n are constants of integration. An infinite series solution of Eq. (22) is then

$$y(\xi,t) = \sum_{n=1}^{\infty} f_n(\xi) g_n(t, A_n, B_n) \tag{36}$$

Equation (36) automatically satisfies boundary conditions (Eqs. 18) at the free end and the root. One might expect that the constants (A_n, B_n) can be chosen to satisfy the conditions at t = 0 and t = t_1; i.e.,

$$\sum_{n=1}^{\infty} f_n(\xi)g_n(0, A_n, B_n) = F_0(\xi)$$

$$\sum_{n=1}^{\infty} f_n(\xi)g_n(t_1, A_n, B_n) = F_1(\xi) \tag{37}$$

Alternatively, the constants A_n, B_n might be chosen so that the beam has a given initial deformation (the first of Eqs. 37) and given initial velocities; i.e., $y_t(\xi,0)$ might be specified instead of $y(\xi,t_1)$.

If $L(t)$ is a sufficiently slowly varying function of t, the approximate solution of Eq. (34) is

$$g_n = A_n \sin\omega_n t + B_n \cos\omega_n t \tag{38}$$

Then, Eq. (36) becomes

$$y(\xi,t) = \sum_{n=1}^{\infty} (A_n \sin\omega_n t + B_n \cos\omega_n t)\, f_n(\xi) \tag{39}$$

The initial value problem then reduces to

$$\sum_{n=1}^{\infty} B_n f_n(\xi) = F_o(\xi)$$

$$\sum_{n=1}^{\infty} A_n \omega_n^{\,o} f_n(\xi) = \phi_o(\xi) \tag{40}$$

in which $\phi_o(\xi) = y_t(\xi,0)$, and $\omega_n^{\,o}$ is the initial value of ω_n. Under fairly broad conditions, the constants A_n, B_n can be chosen to satisfy Eq. (40) when $F_o(\xi)$ and $\phi_o(\xi)$ are prescribed functions. Then Eq. (39) represents the solution of the initial value problem.

The kinetic energy and the strain energy are, respectively,

$$T = \frac{1}{2}\rho L\int_0^1 y_t^{\,2} d\xi = \frac{1}{2}\rho L\sum_{n=1}^{\infty} \omega_n^{\,2}(A_n \cos\omega_n t - B_n \sin\omega_n t)^2$$

$$U = \frac{EI}{2L^3}\int_0^1 y_{\xi\xi}^2 d\xi = \frac{EI}{2L^3}\sum_{n=1}^{\infty} \beta_n^{\,4}(A_n \sin\omega_n t + B_n \cos\omega_n t)^2$$

Hence, in view of Eq. (33), the total mechanical energy is

$$T + U = \frac{EI}{2L^3}\sum_{n=1}^{\infty} \beta_n^{\,4}(A_n^{\,2} + B_n^{\,2})$$

Thus, the total mechanical energy is independent of t, provided that A_n and B_n vary as $L^{3/2}$. Apparently, for the conditions stated, this requirement ensures conservation of mechanical energy.

(b) F(t) Prescribed. If one prescribes F(t), then Eqs. (17) and (20) subject to Eqs. (18) and (21) must be solved simultaneously in order to determine L(t). The method of solution for both cases L(t) prescribed and F(t) prescribed is outlined in the following section.

Method of Solution

Equations (17) and (20), together with the initial and boundary conditions given by Eqs. (18) and (21) form a system of two nonlinear partial differential equations in the two unknowns L(t) and y(x,t). An approximate numerical solution of these equations can be obtained by a Galerkin finite element formulation.

Thus, in accordance with the Galerkin FEM, we form the approximate solutions of L and y as ϕ and v which are,

$$
\begin{aligned}
L(t) &\cong \phi(t) = \beta(t) \\
y(x,t) &\cong v(x,t) = \underset{\sim}{N}^T(x)\underset{\sim}{\alpha}(t)
\end{aligned}
\tag{41}
$$

where $\underset{\sim}{N}$ are a set of quadratic shape functions with local support. A quadratic shape function is associated with each nodal point of the discretized finite element model. Quadratic interpolation over an element requires each element has three nodal points, say one at each end of the element, and a nodal point in the center of each element.

Figure 4 shows that the odd numbered shape functions N_1, N_3, ..., associated with nodal points at the ends of an element span five nodal points (two elements), while the even numbered shape functions N_2, N_4, ..., span the three nodal points of one element.

In accordance with the Galerkin FEM, we form the residual function for Eqs. (17) and (20) as,

$$
\begin{aligned}
R_1(x,t) = {} & \frac{EI}{\phi^4} v'''' + \rho[v''\xi^2 \frac{\dot{\phi}^2}{\phi^2} + 2v'\xi \frac{\dot{\phi}^2}{\phi^2} \\
& - 2\dot{v}'\xi \frac{\dot{\phi}}{\phi} - v'\xi \frac{\ddot{\phi}}{\phi} + \ddot{v}]
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
R_2(l,t) = {} & \ddot{\phi} + \frac{1}{2\rho L_o} [\frac{EI}{\phi^4} v''^2(l,t) - \rho\{- \frac{\dot{\phi}}{\phi} v'(l,t) + \dot{v}(l,t)\}^2] \\
& - \frac{1}{\rho L_o} F(t)
\end{aligned}
\tag{43}
$$

The finite element equations are obtained by requiring that the residual functions, given by Eqs. (42) and (43), be orthogonal to each of the basis functions. That is,

$$\int_0^1 \underset{\sim}{N} R_1 dx = \underset{\sim}{0} \tag{44}$$

$$\int_0^1 H R_2 dx = 0 \tag{45}$$

where H is the function given by

$$H = 1 \qquad 0 \leq x \leq 1 \tag{46}$$

Substitution of Eq. (41) into (42) and (43) gives

$$R_1(x,t) = \frac{1}{\beta^4}(EI\underset{\sim}{N}''^T\underset{\sim}{\alpha})'' + \rho[\xi^2 \frac{\dot{\beta}^2}{\beta^2} \underset{\sim}{N}''^T\underset{\sim}{\alpha} + 2\xi \frac{\dot{\beta}^2}{\beta^2} \underset{\sim}{N}'^T\underset{\sim}{\alpha}$$
$$- 2\xi \frac{\dot{\beta}}{\beta} \underset{\sim}{N}'^T\dot{\underset{\sim}{\alpha}} - \xi \frac{\ddot{\beta}}{\beta} \underset{\sim}{N}'^T\underset{\sim}{\alpha} + \underset{\sim}{N}^T\ddot{\underset{\sim}{\alpha}}] \tag{47}$$

$$R_2(1,t) = \ddot{\beta} + \frac{1}{2\rho L_o}[\frac{EI}{\beta^4}(\underset{\sim}{N}''^T\underset{\sim}{\alpha})^2\Big|_{\xi=1} - \rho\{-\frac{\dot{\beta}}{\beta}\underset{\sim}{N}'^T\underset{\sim}{\alpha}\Big|_{\xi=1} + \underset{\sim}{N}^T\dot{\underset{\sim}{\alpha}}\Big|_{\xi=1}\}^2]$$
$$- \frac{1}{\rho L_o} F(t) \tag{48}$$

Substitution of Eqs. (47) and (48) into Eqs. (44) and (45) respectively, and performing integration by parts, gives

$$\underset{\sim}{B}T + \int_0^1 EI\underset{\sim}{N}''\underset{\sim}{N}''^T\alpha\xi\underset{\sim}{\alpha} - \frac{\dot{\beta}^2}{\beta^2}\int_0^1 \rho\xi^2\underset{\sim}{N}'\underset{\sim}{N}'^T\alpha\xi\underset{\sim}{\alpha} + 2\frac{\dot{\beta}^2}{\beta^2}\int_0^1 \xi\underset{\sim\sim}{NN}'^T d\xi\underset{\sim}{\alpha}$$
$$- 2\frac{\dot{\beta}}{\beta}\int_0^1 \xi\underset{\sim\sim}{NN}'^T d\xi\dot{\underset{\sim}{\alpha}} - \frac{\ddot{\beta}}{\beta}\int_0^1 \xi\underset{\sim\sim}{NN}'^T d\xi\underset{\sim}{\alpha} + \int_0^1 \underset{\sim\sim}{NN}^T d\xi\ddot{\underset{\sim}{\alpha}} = \underset{\sim}{0} \tag{49}$$

and

$$\ddot{\beta} + \frac{1}{2\rho L_o} \frac{EI}{\beta^4} [(\frac{4}{\ell_e^2} \alpha_{2n-1} - \frac{8}{\ell_e^2} \alpha_{2n} + \frac{4}{\ell_e^2} \alpha_{2n-1})^2$$
$$- \rho\{- \frac{\dot{\beta}}{\beta}(\frac{1}{\ell_e} \alpha_{2n-1} - \frac{4}{\ell_e} \alpha_{2n} + \frac{3}{\ell_e} \alpha_{2n+1}) + \alpha_{2n+1}\}^2] = \frac{1}{\rho L_o} F(t) \tag{50}$$

where $\underset{\sim}{BT}$ denotes a boundary term.

Letting,

$$\left.\begin{aligned}
A &= \int_0^1 EI \, \underset{\sim}{N}'' \underset{\sim}{N}''^T \, d\xi \\
B &= \int_0^1 \rho\xi^2 \underset{\sim}{N}' \underset{\sim}{N}'^T \, d\xi \\
C &= \int_0^1 \xi \, \underset{\sim}{N} \, \underset{\sim}{N}'^T \, d\xi \\
D &= \int_0^1 \underset{\sim}{N} \, \underset{\sim}{N}^T \, d\xi
\end{aligned}\right\} \tag{51}$$

$$a = (\frac{4}{\ell e^2} \alpha_{2n-1} - \frac{8}{\ell_e^2} \alpha_{2n} + \frac{4}{\ell e^2} \alpha_{2n+1})^2$$

$$b = \{\frac{\dot{\beta}}{\beta} (\frac{1}{\ell_e} \alpha_{2n-1} - \frac{4}{\ell_e} \alpha_{2n} + \frac{3}{\ell_e} \alpha_{2n+1}) + \dot{\alpha}_{2n+1}\}^2$$

Equations (49) and (51) become

$$\underset{\sim}{BT} + A\underset{\sim}{\alpha} - \frac{\dot{\beta}^2}{\beta^2} A\underset{\sim}{\alpha} + 2\frac{\dot{\beta}^2}{\beta^2} C\underset{\sim}{\alpha} - 2\frac{\ddot{\beta}}{\beta} C\underset{\sim}{\dot{\alpha}} + D\underset{\sim}{\ddot{\alpha}} = \underset{\sim}{0} \tag{52}$$

$$\ddot{\beta} + \frac{1}{2\rho L_o}\frac{EI}{\beta^4}[a - \rho b] = \frac{1}{\rho L_o} F(t) \tag{53}$$

Defining the following terms,

$$\left.\begin{aligned}
\underset{\sim}{\gamma} &= -(\underset{\sim}{BT} + A\underset{\sim}{\alpha}) \\
\underset{\sim}{\omega} &= -\frac{1}{\beta^2}\dot{\beta}^* A\underset{\sim}{\alpha} + 2\frac{\dot{\beta}^*}{\beta^2} C\underset{\sim}{\alpha} \\
C^* &= -2\frac{\dot{\beta}^*}{\beta} C \\
\underset{\sim}{\varepsilon} &= -\frac{1}{\beta} C\underset{\sim}{\alpha} \\
G &= \frac{1}{\rho L_o} F(t) - \frac{1}{2\rho L_o}(a - \rho b)
\end{aligned}\right\} \tag{54}$$

where $\dot{\beta}^*$ is the value of $\dot{\beta}$ at the previous time, Eqs. (52) and (53) become,

$$\begin{aligned}
D\underset{\sim}{\ddot{\alpha}} + \underset{\sim}{\varepsilon}\ddot{\beta} + C^*\underset{\sim}{\dot{\alpha}} + \underset{\sim}{\omega}\dot{\beta} &= \underset{\sim}{\gamma} \\
\ddot{\beta} &= G
\end{aligned} \tag{55}$$

or, in matrix form

$$\begin{array}{r} \\ \\ 2n+1 \\ 2n+2 \end{array}
\begin{array}{c} 1 \qquad\qquad 2n+1 \quad 2n+2 \\ \left[\begin{array}{ccc:c} & D & & \underset{\sim}{\varepsilon} \\ \hdashline 0 & \cdots & 0 & 1 \end{array}\right] \end{array}
\left\{\begin{array}{c} \underset{\sim}{\ddot{\alpha}} \\ \hdashline \ddot{\beta} \end{array}\right\} + \left[\begin{array}{ccc:c} & C & & \underset{\sim}{\omega} \\ \hdashline 0 & \cdots & 0 & 0 \end{array}\right]\left\{\begin{array}{c} \underset{\sim}{\dot{\alpha}} \\ \hdashline \dot{\beta} \end{array}\right\} = \left\{\begin{array}{c} \underset{\sim}{\gamma} \\ \hdashline G \end{array}\right\} \tag{56}$$

Equations (55) (or (56)), subject to initial conditions may be solved by numerical integration.

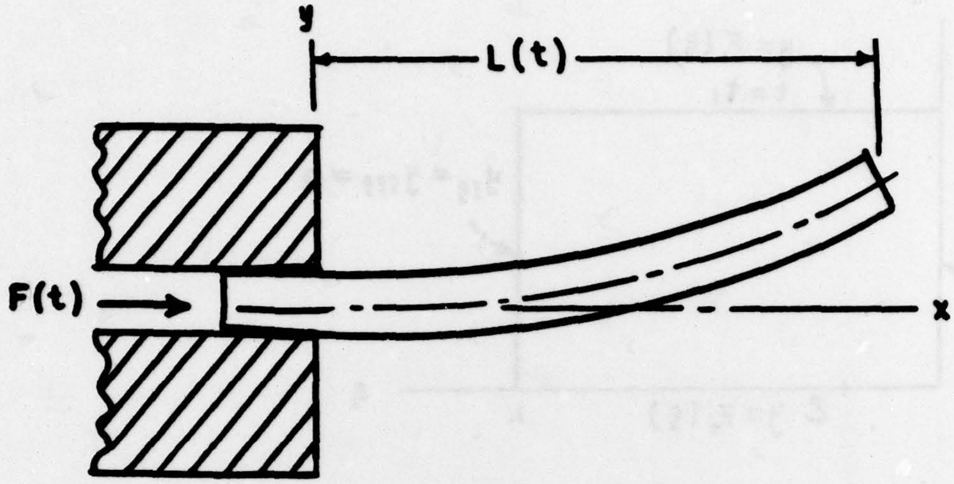

Figure 1. Cantilever Beam Moving Axially in Cylindrical Hole.

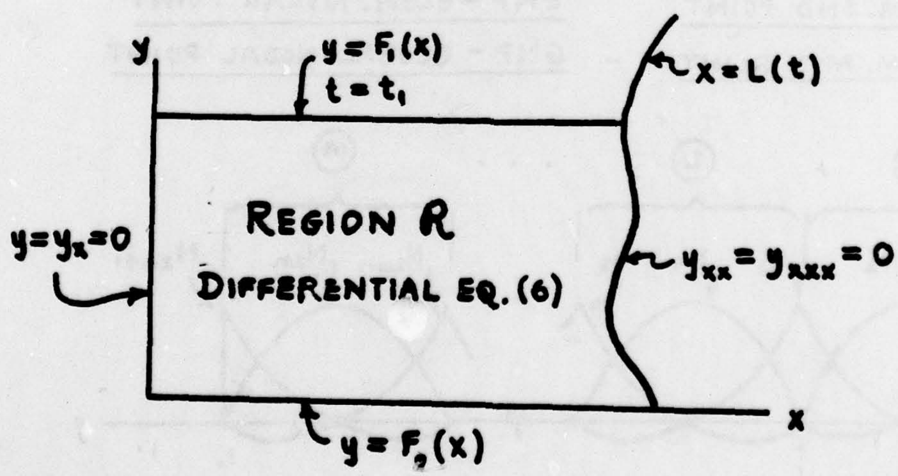

Figure 2. Region of Integration for Equation (6).

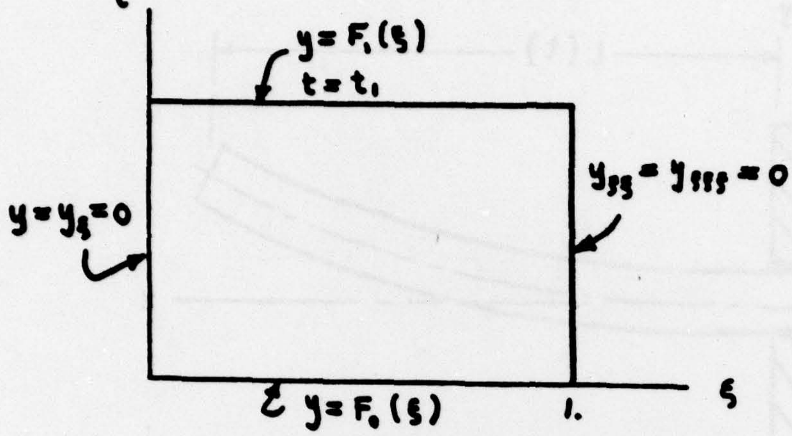

Figure 3. Region of Integration for Equation (17)

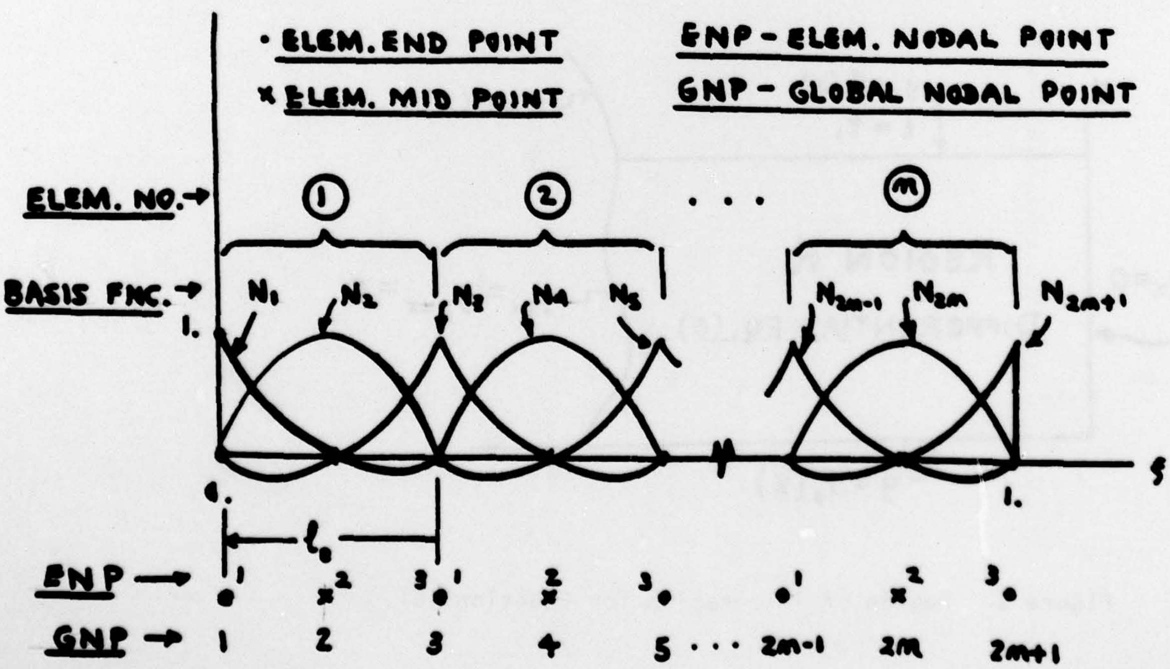

Figure 4. Discretized FEM Model with Quadratic Basis Functions

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